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BAHADUR EFFICIENT TEST FOR THE PARAMETERS OF INVERSE GAUSSIAN DISTRIBUTION

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SUMMARY

Consider the two parameter inverse Gaussian distribution. Suppose one is interested in a testing problem in which the null hypothesis specifies both the parameters. For this problem, a test is proposed based on Fisher's method of combining independent tests. It is shown that the test is asymptotically optimal in the sense of Bahadur efficiency.

Keywords: Random variables; uniformly most powerful unbiased test; Chi-square distribution; strictly increasing function.

1. Introduction

Let $X_1, X_2, \ldots, X_n$ be n independent and identically distributed random variables (i.i.d. r.v.s.) each having the probability density function (p.d.f.)

$$f(x \mid \theta) = \left(\frac{\lambda}{2\pi x^3}\right)^{1/2} \cdot \exp\left[-\lambda\mu^{-2} \cdot x^{-1}(x-\mu)^2/2\right],\ x > 0 \tag{1}$$

where $\theta = (\mu, \lambda)$, $\Theta = \{\theta \mid 0 < \mu < \infty, 0 < \lambda \leqslant \lambda_0\}$, λ_0 specified.

The problem is to test

$$H_0 : \theta \in \Theta_0 \quad \text{against } H_1 : \theta \in \Theta - \Theta_0,$$

where $\Theta_0 = \{\theta \mid \mu = \mu_0, \lambda = \lambda_0\}$, μ_0 specified.

We first propose two tests and combine them by Fisher's method. The new test thus obtained is shown to be Bahadur efficient. For the definition of Bahadur efficiency and related results, refer Bahadur [1]. Let

$$U_n = n \cdot \lambda_0 \cdot \mu_0^{-2} \cdot \bar{X}^{-1} (X - \mu_0)^2$$

$$V_n = \lambda_0 \sum_{i=1}^{n} (X_i^{-1} - \bar{X}^{-1}) \quad \text{where}$$

$$\bar{X} = \sum_{i=1}^{n} X_i/n.$$

For testing $H_0 : \mu = \mu_0$ against $H_1' : \mu \neq \mu_0$, when λ is unknown, Chhikara and Folks [2] have shown that the uniformly most powerful (UMP) unbiased test is based on the test statistic $[1 - (1/n)] U_n/V_n$. However, we propose to take U_n as the test statistic for testing H_0' against H_1'.

We can easily show that for testing $H_0'' : \lambda = \lambda_0$ against $H_1'' : \lambda < \lambda_0$, when μ is unknown, the UMP test is based on the test statistic V_n.

It is known that U_n and V_n are independent and under H_0, V_n has chi-square distribution with $(n - 1)$ degrees of freedom, U_n has a chi-square distribution with one degree of freedom (Tweedie [7] and Shuster [6]). Let

$$\theta_n = -2 \log P_{H_0} [U_n > u] - 2 \log P_{H_0} [V_n > v],$$

where u and v are the observed values of U_n and V_n, respectively.

According to the Fisher's method of combining independent tests, the new test for testing for the original null hypothesis H_0 against the alternative H_1 is given by the critical region $\theta_n \geqslant C$, where C is chosen to satisfy the size condition. Since U_n and V_n have chi-square distributions under H_0, Q_n can be computed easily using chi-square table. Also, it is well known that under H_0, Q_n has chi-square distribution with four degrees of freedom. Thus the cut off point C can also be obtained easily.

2. Bahadur Efficiency of the Test

We now show that the test based on Q_n is Bahadur efficient in that the exact slope, $C_{Q_n}(\theta)$, of the sequence of tests $\{Q_n\}$ is equal to

$$E_\theta \left[\log \left\{ \frac{f(x \mid \theta)}{f(x \mid \theta_0)} \right\} \right] \forall\ \theta \in \Theta - \Theta_0, \tag{2}$$

where $f(x \mid \theta)$ is given by the Equation (1). The following lemmas lead to the computation of $C_{Q_n}(\theta)$.

LEMMA 1. *The exact slope of the sequence of tests $\{U_n\}$ is given by*
$C_1(\theta) = \lambda_0 \cdot \mu^{-1} (1 - \mu\mu_0^{-1})^2 \ \forall\ \theta \in \Theta - \Theta_0$

Proof. Define $U_n^* = U_n/\sqrt{n}$. Then we can easily show that

$$\frac{U_n^*}{\sqrt{n}} \xrightarrow{a \cdot s(\theta)} \lambda_0 \mu^{-1} (1 - \mu\mu_0^{-1})^2 \ \forall\ \theta \in \Theta - \Theta_0$$

Further,

$$\lim_{n \to \infty} \{-1/n \log P_{H_0} [U_n^* \geqslant \sqrt{n}\ \ t]\} = \tfrac{1}{2} \cdot t$$

(see Lemma 1 of the Appendix). Therefore the exact slope of $\{U_n^*\}$ is given by

$$C_1^*(\theta) = \lambda_0 \mu^{-1} (1 - \mu \cdot \mu_0^{-1})^2.$$

Since U_n^* is a strictly increasing function of U_n, $\{U_n\}$ and $\{U_n^*\}$ have the same exact slope which completes the proof.

LEMMA 2. *The exact slope of* $\{V_n\}$ *is given by*

$$C_2(\theta) = \lambda_0 \cdot \lambda^{-1} - 1 - \log (\lambda_0 \cdot \lambda^{-1}) \ \forall\ \theta \in \Theta - \Theta_0$$

Proof. Define $V_n^* = V_n/\sqrt{n}$. We can show that

$$\frac{V_n^*}{\sqrt{n}} \xrightarrow{a \cdot s(\theta)} \lambda_0 \cdot \lambda^{-1} \ \forall\ \theta \in \Theta - \Theta_0$$

Further,

$$\lim_{n \to \infty} \{-1/n \log P_{H_0} [V_n^* \geqslant \sqrt{n} \cdot t]\} = \tfrac{1}{2} [t - 1 - \log t]$$

(see Lemma 2 of the Appendix). Therefore, the exact slope of $\{V_n^*\}$ is given by

$$C_2^*(\theta) = \lambda_0 \cdot \lambda^{-1} - 1 - \log (\lambda_0 \cdot \lambda^{-1}).$$

Since V_n^* is a strictly increasing function of V_n, $\{V_n\}$ and $\{V^*\}$ have the same exact slope which completes the proof.

For details of proof of the above lemmas, see the Appendix and also Perng [5].

LEMMA 3. *The exact slope of* $\{Q_n\}$ *is given by*

$$C_{Q_n}(\theta) = C_1(\theta) + C_2(\theta), \tag{i}$$

and hence

$$C_{Q_n}(\theta) = \lambda_0 \cdot \mu^{-1} (1 - \mu \cdot \mu_0^{-1})^2 + \lambda_0 \cdot \lambda^{-1} - 1 - \log (\lambda_0 \cdot \lambda^{-1}) \ \forall\ \theta \in \Theta - \Theta_0. \tag{ii}$$

For the proof of the first part, see Littell and Folks [4] and the second part follows from the previous lemmas.

THEOREM. *The test based on Q_n is Bahadur efficient.*

Proof. Consider $\forall\ \theta \in \Theta - \Theta_0$

$$\log\left[\frac{f(x \mid \theta)}{f(x \mid \theta_0)}\right] = \frac{1}{2} \log(\lambda \cdot \lambda_0^{-1}) + \frac{1}{2} [\lambda_0 \cdot \mu_0^{-2} \cdot X^{-1} (X - \mu_0)^2 - \lambda \mu^{-2} X^{-1} (X - \mu)^2]$$

Therefore, we can easily see that

$$E_\theta \left\{ \log\left[\frac{f(x \mid \theta)}{f(x \mid \theta_0)}\right]\right\} = \lambda_0 \mu^{-1} (1 - \mu \cdot \mu^{-1})^2 + \lambda_0 \cdot \lambda^{-1} - 1 - \log(\lambda_0 \cdot \lambda^{-1}) = C_{Q_n}(\theta)\ \forall\ \theta \in \Theta - \Theta_0, \quad (3)$$

which completes the proof.

3. Concluding Remarks

The test based on Q_n is thus Bahadur efficient and easy to compute. Moreover, it seems that the Fisher's method considered in this note may yield Bahadur efficient tests for other distributions also, if one chooses the component test statistics suitably. For example, Durairajan [3] has shown that the method yields Bahadur efficient test for the normal distribution with unknown mean and unknown variance.

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APPENDIX

We need the following two well known formulas:

$$\int_u^\infty x^{v-1} \cdot e^{-\mu x} \cdot dx = \mu^{-1} (u)^{v-1} \cdot e^{-\mu u} \cdot \left[1 + O\left(\frac{1}{u}\right)\right] \qquad \text{(i)}$$

for large u; and

$$\log \Gamma(x) = \left(x - \frac{1}{2}\right) \log x - x + \frac{1}{2} \log (2\pi) + O\left(\frac{1}{x}\right) \text{ for large } x. \qquad \text{(ii)}$$

LEMMA 1. *If U_n has a chi-square distribution with one degree of freedom, then*

$$\lim_{n\to\infty} \left\{-\frac{1}{n} \log p\,[U_n \geqslant nt]\right\} = t/2$$

Proof. using the formula (i) given above, we have

$$P\,[U_n \geqslant nt] = \frac{1}{\sqrt{2\pi}} \int_{nt}^\infty e^{-x/2} \cdot x^{\frac{1}{2}-1} \cdot dx$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{2}\right)^{-1} (nt)^{\frac{1}{2}-1} \cdot e^{-nt/2} \left[1 + O\left(\frac{1}{n}\right)\right]$$

for large n and fixed t. Therefore, we have

$$-\frac{1}{n} \cdot \log p\,[U_n \geqslant nt] = \frac{C}{n} + \frac{\log n}{2n} + \frac{t}{2} - \frac{1}{n} \cdot \log\left[1 + O\left(\frac{1}{n}\right)\right],$$

where C is a constant independent of n. Hence the lemma.

LEMMA 2. *If V_n has a chi-square distribution with $(n-1)$ degrees of freedom, then*

$$\lim_{n\to\infty} \left\{\frac{-1}{n} \cdot \log P\,[V_n \geqslant nt]\right\} = \frac{1}{2}\,[t - 1 - \log t]$$

Proof. Using the formula (i), we have

$$P\,[V_n \geqslant nt] = 2^{\frac{1}{\frac{n-1}{2}}}\, \Gamma\left(\frac{n-1}{2}\right) \int_{nt}^\infty e^{-x/2} \cdot x^{\frac{n-1}{2}-1} \cdot dx$$

$$= \frac{1}{2^{n-1/2}} \cdot \frac{1}{\Gamma\frac{n-1}{2}} \cdot \left(\frac{1}{2}\right)^{-1} (nt)^{\frac{n-1}{2}-1} \cdot e^{-nt/2} \left[1 + O\left(\frac{1}{n}\right)\right]$$

for large n and fixed t. Therefore,

$$-\log P[V_n \geqslant nt] = \log \Gamma\left(\frac{n-1}{2}\right) - \left(\frac{n-3}{2}\right)\log\left(\frac{n}{2}\right)$$
$$-\left(\frac{n-3}{2}\right)\log t + \frac{nt}{2} - \log\left[1 + O\left(\frac{1}{n}\right)\right].$$

Using formula (ii), we get

$$-\frac{1}{n}\log p[V_n \geqslant nt] = \frac{1}{2}\left[\log\left(1-\frac{1}{n}\right) - \frac{2}{n}\log\left(\frac{n-1}{2}\right)\right.$$
$$\left.+\frac{3}{n}\log\left(\frac{n}{2}\right)\right] - \left(\frac{1}{2} - \frac{1}{2n}\right) + \frac{\log(2\pi)}{2n} - \left(1 - \frac{3}{n}\right)\frac{\log t}{2}$$
$$+\frac{t}{2} + O\left(\frac{1}{n}\right) - \frac{1}{n}\log\left[1 + O\left(\frac{1}{n}\right)\right],$$

which completes the proof.